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Analysis of random nonlinear water waves: the Stokes–Woodward technique

La technique de Stokes–Woodward pour l'analyse de vagues aléatoires non linéaires

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Abstract

A generalization of the Woodward's theorem is applied to the case of random signals jointly modulated in amplitude and frequency. This yields the signal spectrum and a rather robust estimate of the bispectrum. Furthermore, higher order statistics that quantify the amount of energy in the signal due to nonlinearities, e.g., wave–wave interaction in the case of water waves, can be inferred. Considering laboratory wind generated water waves, comparisons between the presented generalization and more standard techniques allow to extract the spectral energy due to nonlinear wave–wave interactions. It is shown that our analysis extends the domain of standard spectral estimation techniques from narrow-band to broad-band processes. ***To cite this article: T. Elfouhaily et al., C. R. Mecanique 331 (2003).***

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1. Introduction and issues

According to common observations, a recent study [1] has demonstrated that wind waves cannot be characterized by a deterministic system dynamically affected by nonlinearities.This leads to a description of water waves as a superposition of random waves characterized by highly irregular amplitudes and frequencies.

When simulating nonlinear water waves, [2] demonstrated that a common inconsistency was to use an empirically determined spectrum as input, despite the fact that, due to the short wave energy increase by wave–wave interaction, the output spectrum will deviate considerably from the measurements. Examples of such misuse have already been identified in [2,3] which are concerned with understanding the electromagnetic bias observed by radar altimeters over the ocean surface. The contribution by [2] suggests the need for an input spectrum, termed the "bare" spectrum, devoid of any nonlinear interaction. The output spectrum is then obtained from nonlinear interactions of all the modes present at the input to form the so-called "dressed" spectrum. Unfortunately, the "bare" spectrum does not itself yield easily to measurement since nonlinear wave–wave interactions cannot be turned off during the mea-

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surement of the surface wave spectrum. Higher order hydrodynamic interactions [4] will distort the input spectrum even further hampering the study of its effect on, among other things, several remote sensing parameters [5].

In the narrow-banded spectrum case, according to [6–9], the weak modulations that characterises surface waves are time independent or slowly varying random variables. However, this narrow-band formulation is insufficient to explain the complexity of water waves when higher-order statistics must be included due to the asymmetric behavior caused by nonlinear wave–wave interactions. In this case, large deviations from a narrow-band approximation can be observed especially under conditions of wind generated waves, and the random variables are no longer time independent [10]. Under these conditions, the processes become broad band in nature.

Woodward's theorem [11] shows that a frequency or phase modulated signal has a spectrum expressed by $S(f) \approx \frac{A^2}{2} P_{\dot{\phi}}(f - f_c)$, $P_{\dot{\phi}}$ being the probability density function of the modulating instantaneous frequency, A a constant amplitude, f_c the central or carrier frequency. For simplicity negative frequencies have been folded onto positive frequencies since the signal under study is real.

In this study, we model random nonlinear surface waves as broad-band processes with the objective of properly characterizing the spectral content of these signals. A distinction will be made between spectral density due to nonlinearity as opposed to that when no wave–wave interactions are present. To achieve these goals, we generalize Woodward's theorem [11] to include amplitude modulations under moderately large indices of modulations: in this context, the form of the signal to be studied is then

$$\eta(t) = a\left(\frac{t}{\mu}\right)\cos\left[\omega_c t + 2\pi \int_0^t D\left(\frac{\tau}{\nu}\right)\mathrm{d}\tau\right] \tag{1}$$

where $a(t/\mu)$ is a random process with an index of modulation μ and $D(\tau/\nu)$ is the modulating random instantaneous frequency with index ν. As suggested by [12], assuming large enough modulation indices μ and ν, statistical stationnarity for a and D, and following [13], we obtain the corresponding spectrum (where $\omega = 2\pi f$):

$$S(f) \approx \frac{1}{2}\int \left[a^2 - \frac{\dot{a}^2}{4\mu^2}\left(\frac{\partial^2}{\partial\omega^2}\right)\right] P\left(a, \dot{a}, \omega - \frac{\dot{\omega}t}{\nu}, \dot{\omega}\right)\mathrm{d}a\,\mathrm{d}\dot{a}\,\mathrm{d}\dot{\omega} \tag{2}$$

This generalization of Woodward's theorem requires knowledge of the joint distribution of four random processes as opposed to that of one process in the original theorem. This generalization to a four dimensional distribution in (2) is generally not useful since it is impractical to estimate such multidimensional distributions from the time series of the signal itself. We shall now present in the following section, a more practical formulation of this generalization where a joint distribution with fewer dimensions is required. We will then compare our development with experimental data obtained from a wind-wave tank, where a clear difference between the bare and the dressed spectra is shown.

4. Conclusion

A generalization of Woodward's theorem is successfully obtained by including random amplitude modulations in addition to frequency modulations. The original theorem stated that a good approximation of the energy spectrum of a frequency modulated signal is the probability density function of the instantaneous frequencies when the index of modulation is high. Our generalization starts by including the random amplitude modulation which yield a simple spectrum expressed as a single integral over the instantaneous amplitudes and the joint distribution of amplitude and frequency as shown in (3). It is noted that this spectrum is devoid of any nonlinearity or mode coupling because over the scale of a characteristic period, the wave is considered as simply harmonic (a sine wave). Asymmetries in the wave profile must be introduced in order to capture residual energy not explained by the "bare" spectrum. To account for this residual energy, we have proposed a second generalization of Woodward's theorem that utilizes a Stokes-like waveform in which all the parameters are random except the time variable. Our procedure is termed the Stokes–Woodward technique since it combines a generalization of Woodward's theorem with a Stokes-like random wave profile.

The second generalization provides a practical formulation for the "dressed" spectrum where nonlinearities up to the second order are included (9). This second order coupling between modes initiates the existence of the bispectrum which can be formulated as in (10). It is demonstrated that when the Stokes–Woodward technique is applied to a time series of water-wave surface elevations, it discriminates between the "bare" and "dressed" spectrum, and also provides a robust estimate of the bispectrum. We recommend that the bare spectrum be used at the input of nonlinear system simulators as originally cautioned in [2].

Applications of the Stokes–Woodward technique will have great benefit in the analysis of nonlinear random processes present in several science fields. For example, it can readily be applied to remote sensing signals as already demonstrated by [19] even with the original formulation of Woodward's theorem.