An intrinsic stabilization scheme for proper orthogonal decomposition based low-dimensional models

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Despite the temporal and spatial complexity of common fluid flows, model dimensionality can often be greatly reduced while both capturing and illuminating the nonlinear dynamics of the flow. This work follows the methodology of direct numerical simulation (DNS) followed by proper orthogonal decomposition (POD) of temporally sampled DNS data to derive temporal and spatial eigenfunctions. The DNS calculations use Chorin's projection scheme; two-dimensional validation and results are presented for driven cavity and square cylinder wake flows. The flow velocity is expressed as a linear combination of the spatial eigenfunctions with time-dependent coefficients. Galerkin projection of these modes onto the Navier-Stokes equations obtains a dynamical system with quadratic nonlinearity and explicit Reynolds number (Re) dependence. Truncation to retain only the most energetic modes produces a low-dimensional model for the flow at the decomposition Re. We demonstrate that although these low-dimensional models reproduce the flow dynamics, they do so with small errors in amplitude and phase, particularly in their long term dynamics. This is a generic problem with the POD dynamical system procedure and we discuss the schemes that have so far been proposed to alleviate it. We present a new stabilization algorithm, which we term *intrinsic stabilization*, that projects the error onto the POD temporal eigenfunctions, then modifies the dynamical system coefficients to significantly reduce these errors. It requires no additional information other than the POD. The premise that this method can correct the amplitude and phase errors by fine-tuning the dynamical system coefficients is verified. Its effectiveness is demonstrated with low-dimensional dynamical systems for driven cavity flow in the periodic regime, quasiperiodic flow at Re=10000, and the wake flow. While derived in a POD context, the algorithm has broader applicability, as demonstrated with the Lorenz system. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2723149]

I. INTRODUCTION

One route to extracting a higher level of information from numerical or experimental data is to look for the coherent structures in the flow as identified by the proper orthogonal decomposition (POD). This decomposition represents the flow field as a linear combination of spatial and temporal basis functions derived from the statistics of the sampled flow field (snapshots). Moreover, it orders the modes by their importance in the flow reconstruction, so that significant data reduction can be achieved by neglecting the least important terms with quantifiable negligible loss in the accuracy of the representation. This process allows one to see the important structures in the flow.

A further step is needed to gain dynamical information from the POD. This can be done by first replacing the flow variables in the Navier-Stokes equations by their POD expansions, leaving the Reynolds number (Re) as a parameter. A Galerkin projection onto the spatial basis functions results in a set of ordinary differential equations representing a dynamical system whose solution at the Reynolds number of simulation is a model of the dynamics of full DNS simulation. It is practical and convenient to truncate this model to obtain a low-dimensional system. The degree to which this system's dynamics reproduce the original time-dependence is a measure of the low-dimensional model's fidelity and usefulness.

It is of further interest for this model to reproduce the dynamics of the full system *away* from the decomposition Reynolds number, for such a valid model would allow predictions of flow behavior in regimes where no DNS has been performed. Thus much more information can be obtained by doing a parameter continuation based on the Reynolds number (Re). This is the key to investigating flow transitions since now they are equivalent to bifurcation phenomena in the dynamical system. However, one impediment to this goal is the failure of the dynamical system to exhibit the correct asymptotic behavior even at the modeled Reynolds number. Holmes *et al.*[1] note several probable causes for this deficiency:

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- loss of the natural energy cascade due to low-dimensional truncation,
- numerical error in the dynamical system coefficients, particularly those involving derivatives,
- neglect of boundary or pressure terms in the computation of the dynamical system coefficients (problem and domain dependent), and
- an incomplete basis as another consequence of truncation which implies that only velocity fields close to the spatial structures of the ensemble average will be reproduced well.

The impact of the last point on the validity of the dynamical system has been addressed by Rempfer,[2] and others have sought to augment the POD spatial basis to attain better representation, e.g., Bangia *et al.*[3] and Jørgensen *et al.*[4] In particular, the unstable steady flow field might not be adequately represented by the POD spatial basis, hence not a solution of the derived dynamical system. Noack *et al.*[5] have shown that the addition of a mean-shift mode is a means of stabilizing the derived dynamical system and compensating for missing phase space directions.

In contrast, we present a means to improve the accuracy and stability of the dynamical system itself without directly addressing the source of the error. This is possible by recognizing that the POD process can provide a low-dimensional representation of the flow that is consistent with DNS, and that it also gives us the correct solution of the derived dynamical system for the initial conditions consistent with the snapshots, the temporal eigenfunctions (unnormalized). This allows us to adjust the computed coefficients so that time integration of the dynamical system does reproduce the correct solution.[6] We do not claim that this is sufficient for all purposes; in particular, it is not sufficient for parameter continuation in Reynolds number in and of itself. However, we demonstrate here that it is sufficient to capture the correct asymptotic behavior exhibited by the DNS. This is a necessary first step before gaining validity over a range of Reynolds numbers.

In this work we systematically go through the POD procedure for some test problems and demonstrate the viability of this new approach to obtain robust low-dimensional models at the Reynolds number of decomposition. Moreover, an example with the Lorenz system shows that this technique may be extended to a dynamical system obtained by Galerkin projection on any set of orthogonal basis functions with appropriate changes in the implementation.

The broad area of POD research has been active since Lumley introduced the application of POD to the study of turbulence in the late 1960s.[7] Coherent structures have long been observed in turbulent flow experiments, such as the Von Kármán vortex street behind a circular cylinder where it originates in the laminar flow regime and persists well into turbulent regime. Numerous papers attest to the success of low-dimensional models based on POD for many fluid flow problems. A comprehensive review of work in this field as well as a complete explanation of the techniques involved can be found in Holmes *et al.*[1]

Finding the optimal basis for a linear decomposition of a data set is relevant to many fields in mathematics and science. The Karhunen-Loève method was initially proposed (independently) by Karhunen[8] in 1946 and Loève[9] in 1955. The method is known by different names depending on the field of study.[10] For example, principal component analysis, proper orthogonal decomposition, empirical eigenfunction decomposition, and singular value decomposition are a few of the alternate names for equivalent procedures. It continues to be a viable topic for research and application.[11,12] More recently, control applications have utilized the POD for the creation of low order models that capture the nonlinear dynamics of the flow.[13,14]

The POD has also been used as a nonlinear dynamics tool applied to nonturbulent flow regimes to extract the spatial and temporal characteristics of the flow. When the POD is applied to a spatiotemporal data set of an evolving flow, it simultaneously derives spatial and temporal orthogonal modes which are coupled. This bi-orthogonality was noted by Sirovich[15] and highlighted by Aubry,[16] and can be mathematically defined as the representation of a flow field $\mathbf{u}(\mathbf{x},t)$ in terms of basis functions $\theta_i(t)$ and $\mathbf{\Phi}_i(\mathbf{x})$ such that

$$\mathbf{u}(\mathbf{x},t) = \sum_i \lambda_i \theta_i(t) \mathbf{\Phi}_i(\mathbf{x}),$$

with

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$$

and

$$\langle \theta_i, \theta_j \rangle = \langle \mathbf{\Phi}_i, \mathbf{\Phi}_j \rangle = \delta_{ij}.$$

Typically, the orthogonality of the temporal modes is ignored since the main objective is the dynamical system based on the spatial modes. However, this property is used to our advantage in the present work.

The first paper to apply these tools to flows in complex geometries was Deane *et al.*[17] That paper constructed low-dimensional models for flow in a periodically grooved channel and for flow past a circular cylinder. Two-dimensional simulations yield a steady flow which gives way to a periodic flow at a critical Reynolds number specific to the problem. Both flows were studied in the periodic regime and both proved amenable to representation via low-dimensional models at the Reynolds number simulated although the long-term dynamics of both systems were found to suffer from some amplitude errors even though the system remained in a stable limit cycle. Also, the four-mode dynamical system for the circular cylinder was unstable, even though four modes captures over 99% of the energy. A far more difficult problem also tackled in the Deane *et al.* paper, however, was predicting the flow properties for a range of Reynolds numbers from the models. They concluded that low-dimensional models of bounded flows such as the grooved channel flow performed better than those of open flows such as the cylinder wake in regimes away from the decomposition Reynolds numbers. The latter was found to be wholly inadequate.

Parameter continuation of these low-dimensional models remains a challenge. This work addresses one of the major obstacles to its success, viz. sufficient accuracy in the low-dimensional model.