Amundsen Sector Response to IPCC Climate Change Model Projection

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We thank the NSF, which has supported the development of this model over many years through several different grants.

- Oral presentations are limited to contributions on the following topics.
  - "Fuzzy Math" - Why aren't our models good enough yet?
  - "Hogging the Limelight" - What are we learning about PIG and the rest of the Amundsen Sea Embayment?
  - "Hey, Over Here!" - What are we missing by focusing on the Amundsen Sea and why should we care?
  - "Working on the Chain Gang" - What are the critical linkages that drive the behavior of the ice sheet?
  - "Lost at Sea" - What have we been missing all these years by ignoring the ice shelves?

14 August email:

- The five focus questions that were posed generated a lot of excellent submissions, however, there was so much cross linking of information between these topics that I **abandoned** them in structuring the agenda.
- Instead, the agenda is organized by the topics: ice shelves and ocean; grounding lines; basal conditions; and ice sheets.
- Within each topic the talks start with observations and end with models. My intention with this structure is to help reveal if the measurements and models are supporting each other. A plenary discussion is included at the end of each topic to address this subject.

Boundary conditions for a full-momentum solver: 1) The dilemma of sliding 2) how do we do embedded models?

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The Full Momentum Solver

- The holy grail of ice sheet modeling.
- In principle,
  - conservation of momentum,
  - coupled with a flow law,
- can provide a differential equation solvable for velocities at every point within the ice sheet volume.

The shallow-ice approximation

- neglects all but the basal drag
- and is useful for slow-moving inland ice.

The shallow-ice approximation

- Only stress allowed is the basal drag.
- Stress assumed to be linear with depth.
- Vertical velocity profile from integrated strain rate.
- Quasi-2D, with Z integrated out.
- One degree of freedom per node.
- Good for interior ice sheet and where longitudinal stresses can be neglected.
- Probably not very good for ice streams.

The Morland-MacAyeal equations

- neglect all but the longitudinal stresses
- and are useful for ice shelves
- and perhaps, in limited circumstances, ice streams.

The Morland-MacAyeal equations

- A modification of the Morland Equations for an ice shelf pioneered by MacAyeal and Hulbe.
- Quasi-2D model (plug flow in X and Y, with Z integrated out).
- Three degrees of freedom (Ux, Uy, and h) vs one (h).
- Addition of friction term violates assumptions of the Morland derivation.
  - Requires specification as to where ice stream occurs.

- These approximations take advantage of the different scales of the horizontal versus the vertical dimensions of the ice sheet,
- and involve an integration and removal of the vertical coordinate.
- Both of these approximations have severe limitations, especially in the dynamically critical ice streams that drain most of the mass out of Antarctica.
- The key interaction of shelf and inland ice, though the ice stream, cannot be adequately captured with either of these "end-member" approximations.

- A full-momentum solver that
  - neglects no stresses and
  - makes no assumptions or vertical integrations
- should give us the best and most accurate model for ice streams.

The computational requirements for such a model are not reasonable for a whole-ice sheet simulation, and hence we have pursued the embedded-grid approach, whereby a shallow-ice model is run for the whole ice sheet, and the full-momentum solver is applied only to a sub-region where ice stream dynamics are known to be important.

- As such there are three different types of boundary conditions that must be specified,
  - the top, the bottom, and the sides.
- The top is easy, a free-boundary is easily specified.
  - Simply no constraits.
- The sides and bottom are more difficult.

- On the sides we have a choice of boundary condition type. Either:
  - Dirichlet: specified boundary velocities (the unknowns, or degrees of freedom in the full-momentum solver)
  - Neumann: specified pressures or surface tractions (the source of momentum).

- From the shallow-ice model in which the full momentum solver is embedded, we can provide both of these conditions,

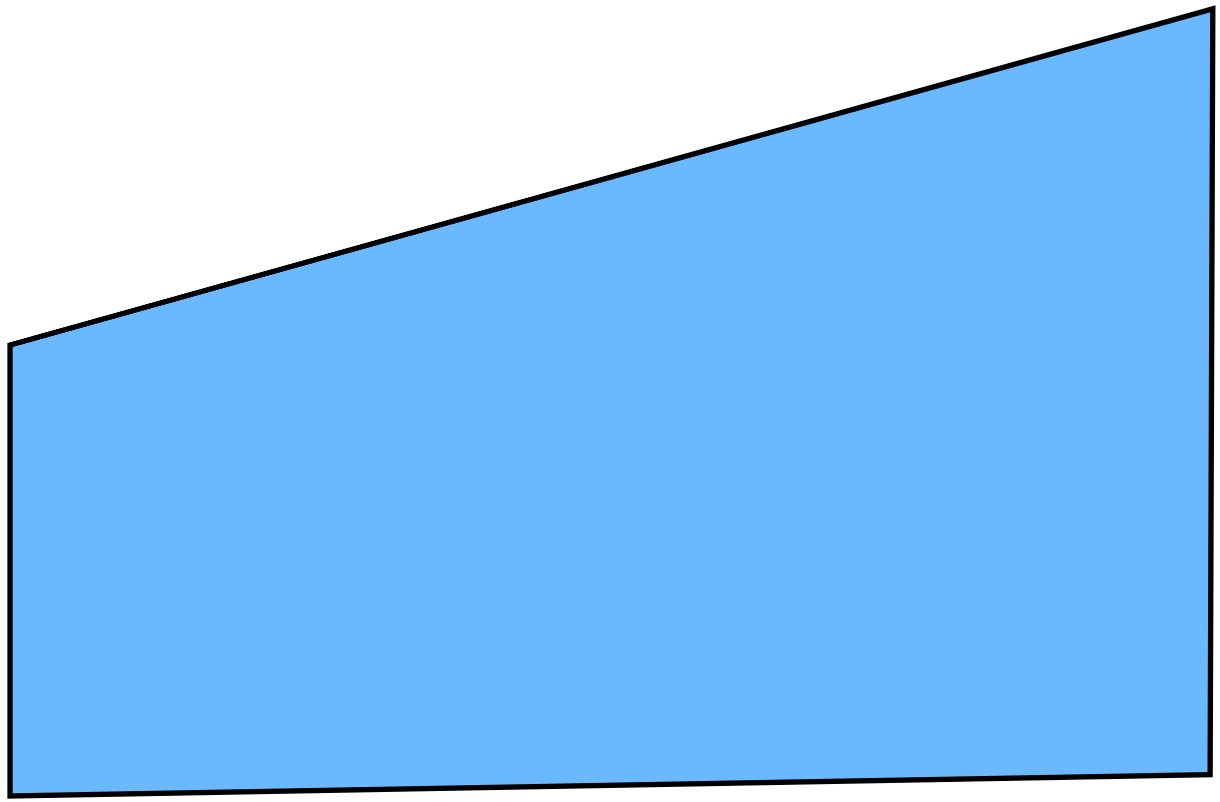
    - although for Dirichlet, the vertical variation in velocity is only of lower order.

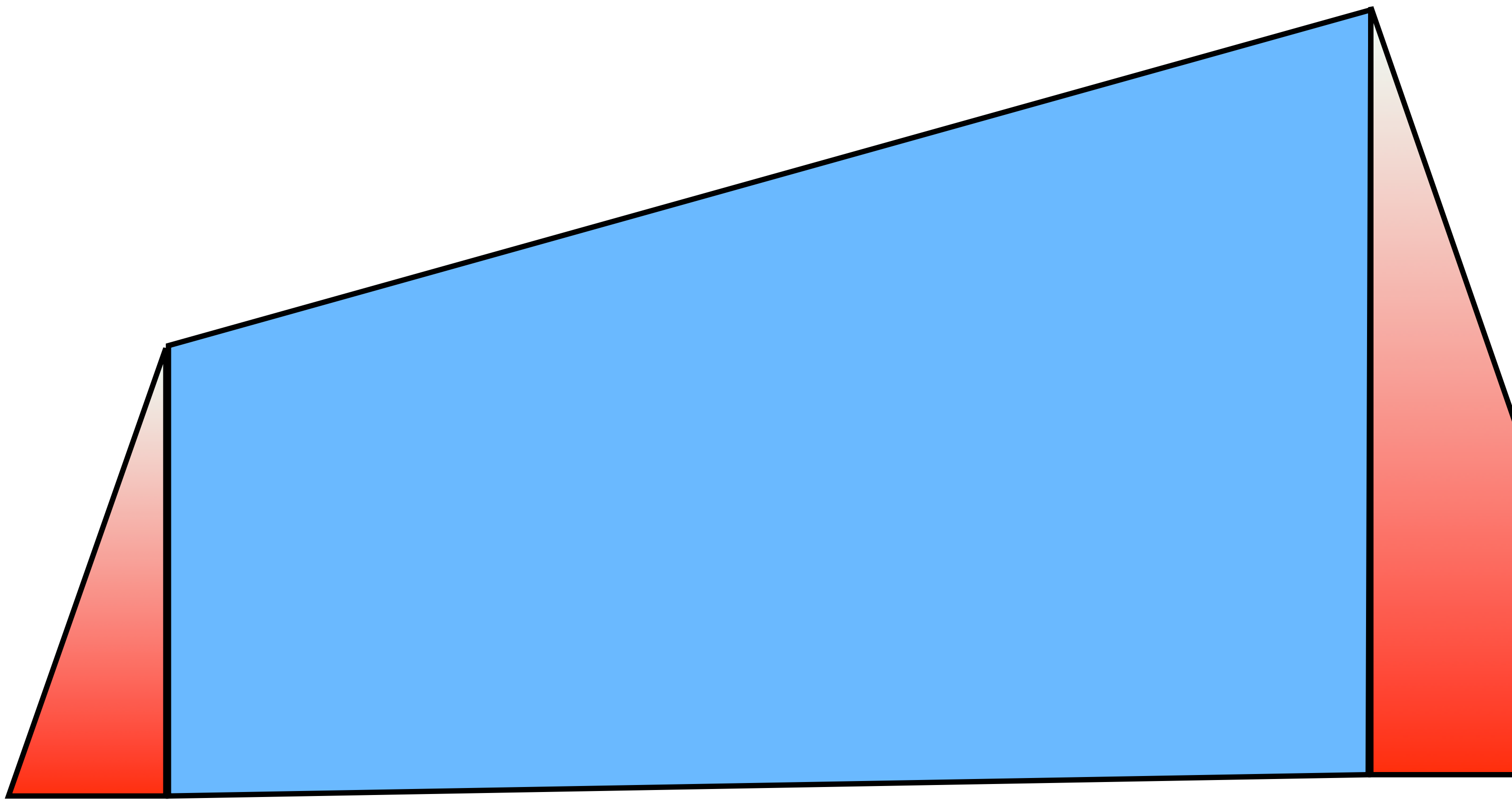
        (Numerical integration of linearly varying driving stress through the temperature-dependent flow law)

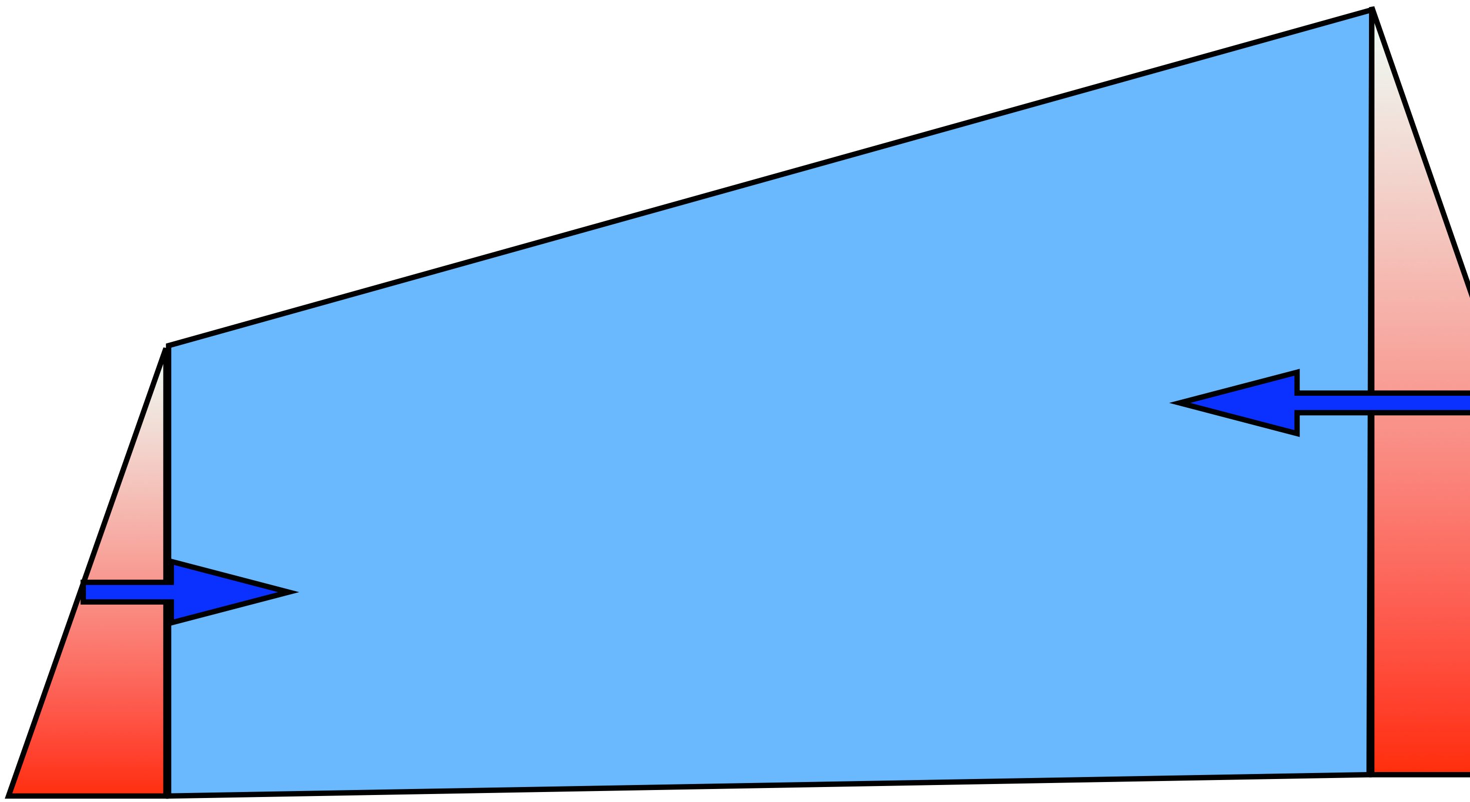
- For Neumann, pressures are not difficult to specify (a simple function of depth).
- However, conservation of angular momentum

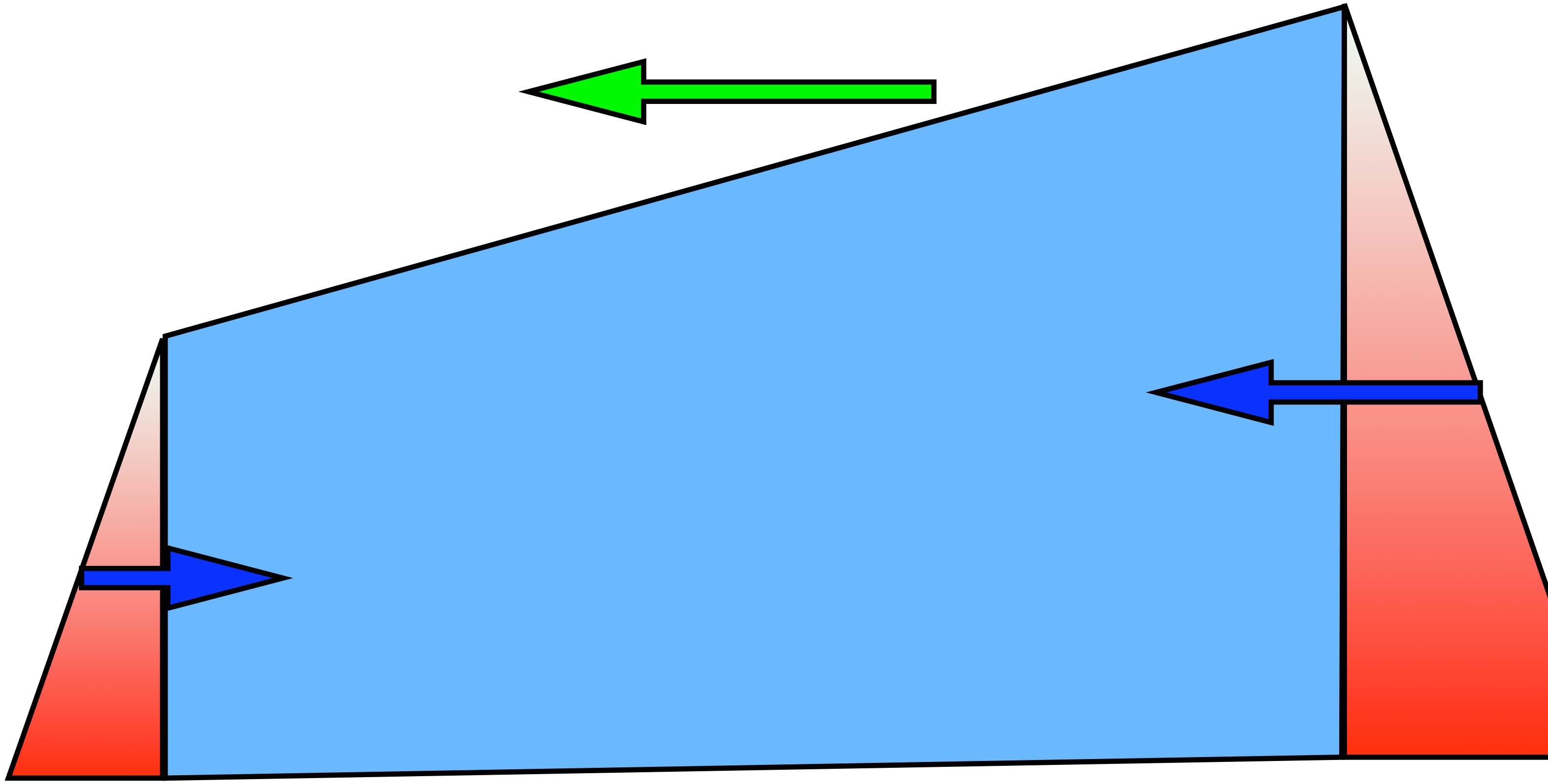
  (net rotational torque must be zero),
- does require specification of some surface traction

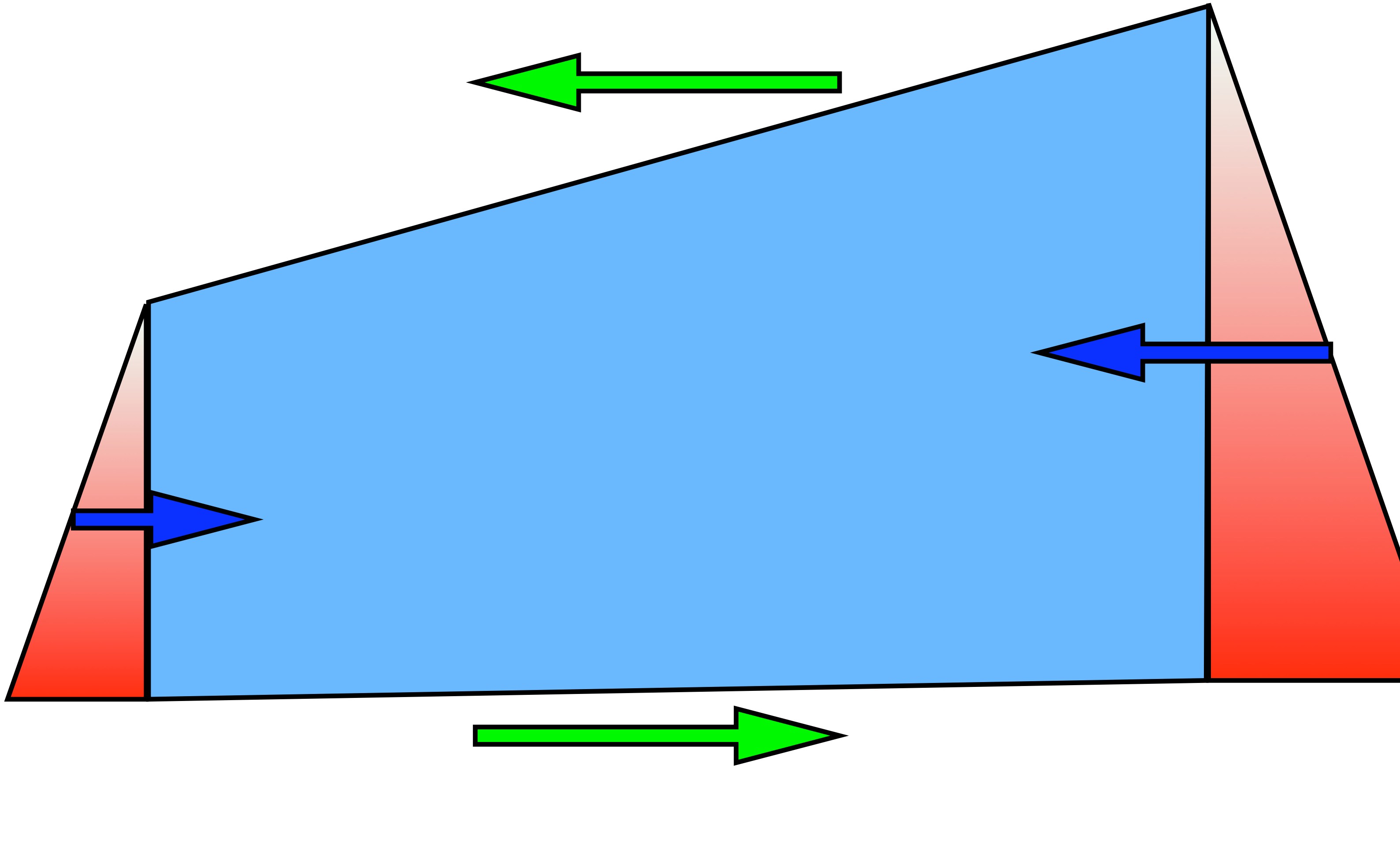
  (the "dynamical stresses"),
- and this can be problematic.

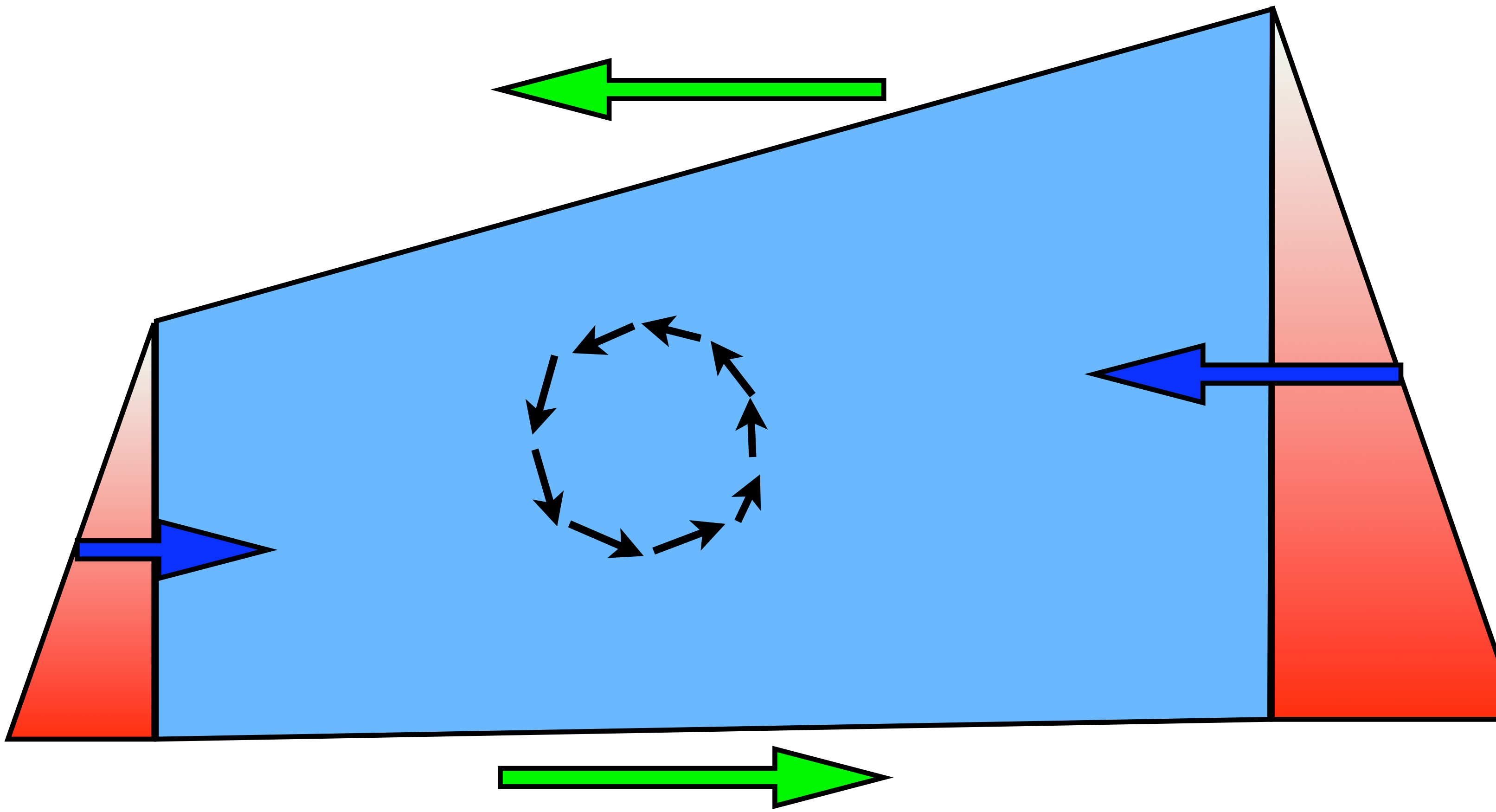


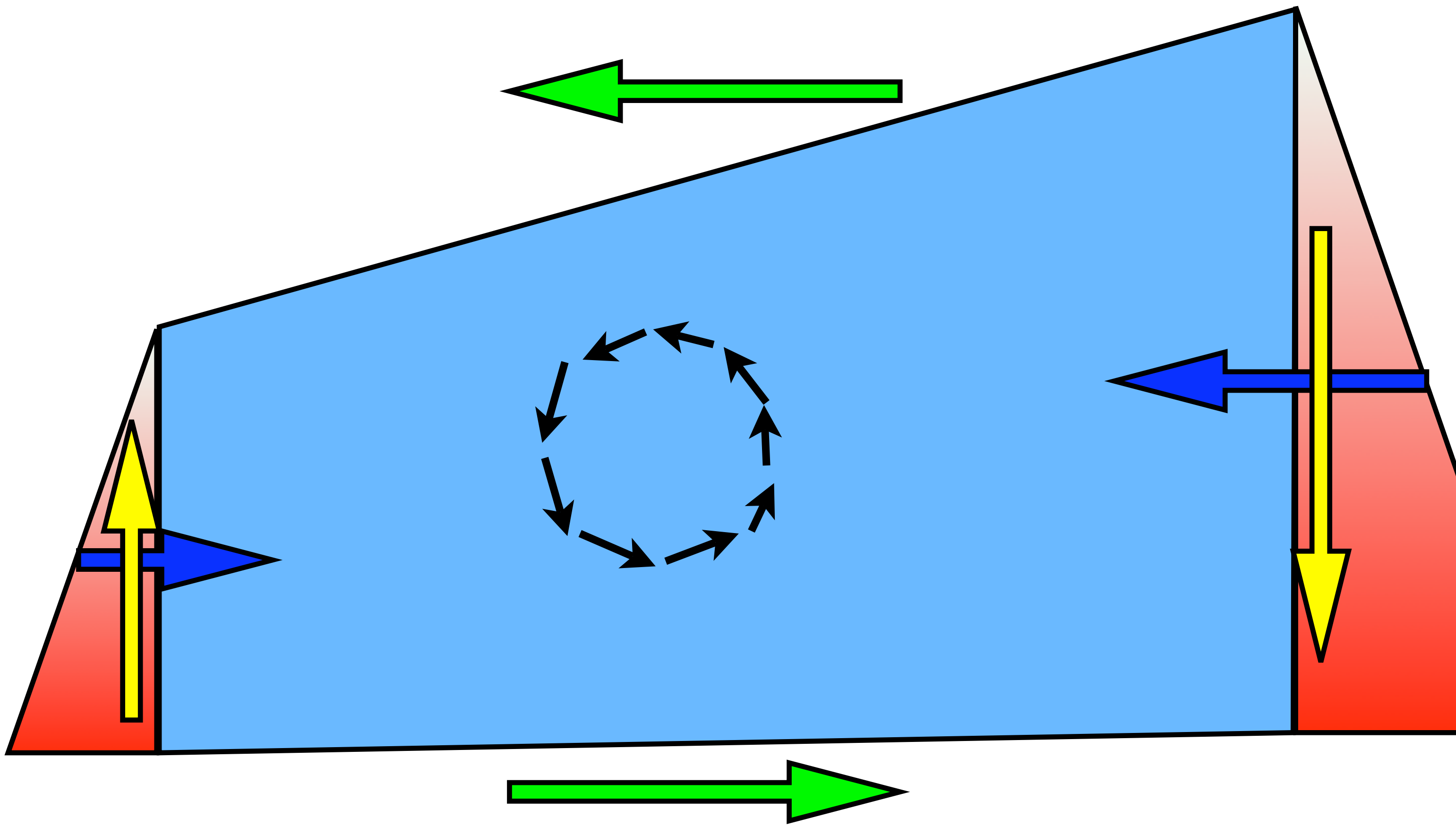


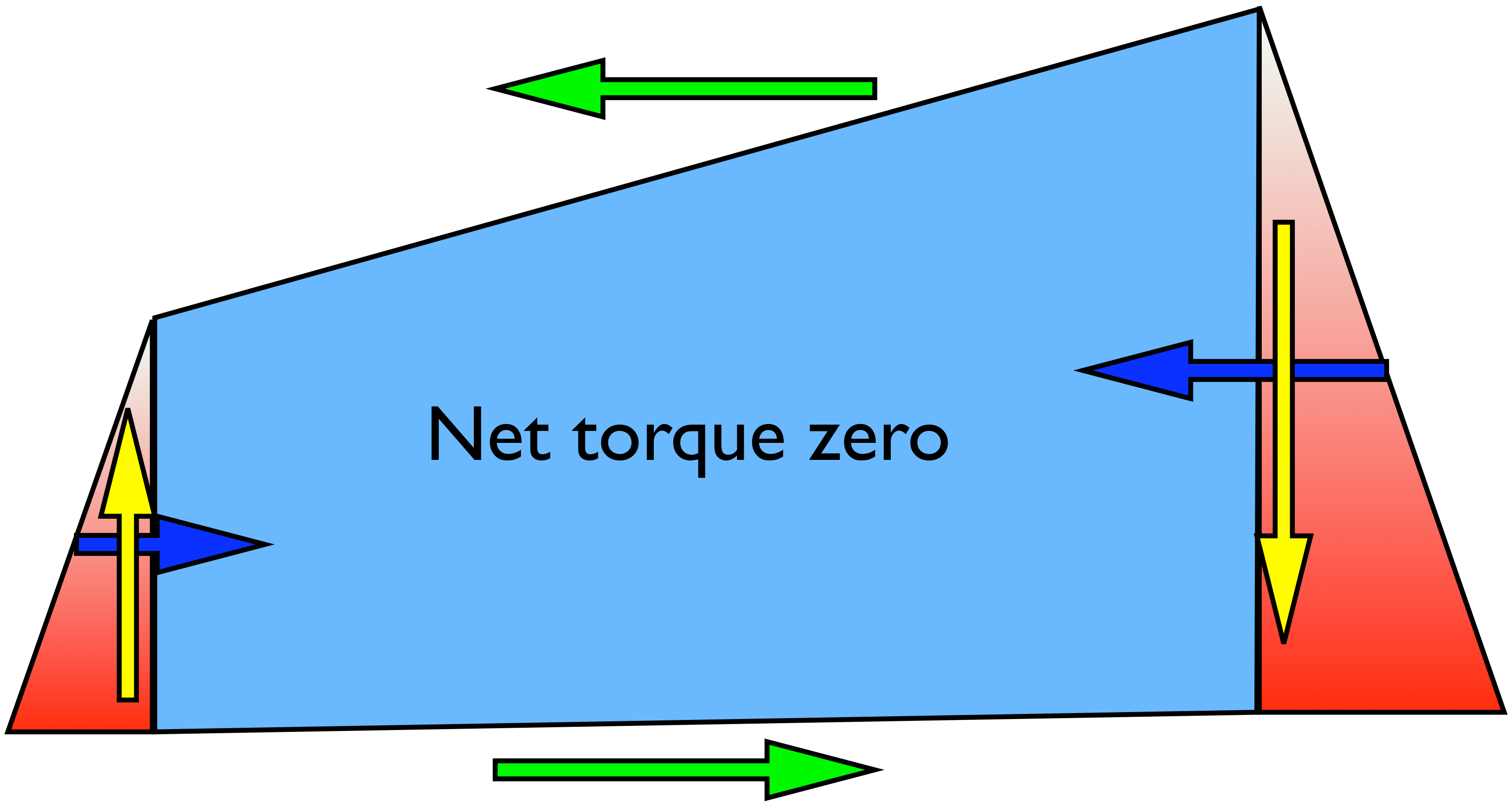











Net torque zero

- Specification of the bottom boundary is more difficult, due to the poorly understood nature of sliding.

  (hard rock, deformable sediments, polythermal ice, basal water, etc.).

- A frozen bed is easy, a simple Dirichlet BC with all velocities specified at zero.
- A completely uncoupled bed is also easy, with a simple free BC in the two horizontal dimensions.

  (In ALL of these cases the vertical velocity is specified to be zero, although it could be the basal melt/freeze rate)

- With Neumann boundary conditions, we can specify the basal traction.
- If this is specified to be equal to the driving stress

  (rho*g*h*alpha),

- we obtain the same solution that we get when we specify no sliding

  (Dirichlet, all velocities zero).

- In reality, the basal stress should be less than the driving stress, with some portion taken up by side shear and longitudinal stresses.

- We have tried specifying a given fraction of the driving stress, but this leads to unrealistic oscillations in the ice sheet profile.

- A uniform stress works well, but there is no indication that this is a reasonable assumption, nor does this deal well with the transition from inland to streaming to shelf.

- Both of these also require "yet-another-parameter," and hence are undesirable.

- A third approach involves a " deformable" basal layer, (a thin layer of elements, the order of meters thick, which is much softer.

- With this approach, one can preserve the easy-to-implement Dirichlet boundary conditions of all basal velocities specified at zero, and still obtain high sliding velocities, and plug-like flow.

- The dilemma is of course the requirement of a "parameter" (how soft and how thick is this layer?)

- Tuning such a model (and remember, this is how the parameters in most sliding laws are obtained, by tuning) would involve comparison of measured and modeled velocity fields in well-documented areas such as the Siple Coast, and soon the Amundsen Sector.

THANK YOU

- Supplemental material

Einstein Notation

- The convention is that any repeated subscript implies a summation over its appropriate range.

- A comma implies partial differentiation with respect to the appropriate coordinate.

$$u_{i,i} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \tag{1}$$

$$\sigma_{ij,j} + \rho a_i = 0$$

$$\begin{aligned}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + \rho a_x = 0 \\
\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + \rho a_y = 0 \\
\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho a_z = 0
\end{aligned} \tag{2}$$

The Full Momentum Equation

- Conservation of Momentum: Balance of Forces

$$\sigma_{ij,j} + \rho a_i = 0 \tag{1}$$

- Flow Law, relating stress and strain rates.

$$\sigma_{ij} = \delta_{ij} P + 2\mu\dot{\epsilon}_{ij} \tag{2}$$

- Effective viscosity, a function of the strain invariant.

$$2\mu = B\dot{\epsilon}^{\frac{1-n}{n}} \tag{3}$$

The Full Momentum Equation

- The strain invariant.

$$\dot{\epsilon}^2 = \frac{1}{2}\dot{\epsilon}_{ij}\dot{\epsilon}_{ij} \tag{4}$$

- Strain rates and velocity gradients.

$$\dot{\epsilon}_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{5}$$

- The differential equation from combining the conservation law and the flow law.

$$(\delta_{ij}P + 2\mu\frac{1}{2}(u_{i,j} + u_{j,i}))_{,j} + \rho a_i = 0 \tag{6}$$

The Full Momentum Equation

- FEM converts differential equation to matrix equation.

$$(K_{mn} + K'_{mn})U_n = F_m \tag{7}$$

- K_{mn} as integral of strain rate term.

$$K_{mn} = \iiint \Psi_{mi,j}\mu(\Psi_{il,j} + \Psi_{jl,i})\, dV \tag{8}$$

- Shape functions as linear FEM interpolating functions.

The Full Momentum Equation

- Elimination of pressure degree of freedom by Penalty Method.

$$P \tilde{=} \Lambda U_{i,i} \quad (9)$$

- K'_{mn} as integral of the pressure term.

$$K'_{mn} = \iiint \Psi_{mi,i} \Lambda \Psi_{jl,j} \, dV \quad (10)$$

- Load vector, RHS, as integral of the body force term.

$$F_m = - \iiint \Psi_{mi} \rho a_i \, dV \quad (11)$$

$$[v_i\{\delta_{ij}P + \mu(u_{i,j} + u_{j,i})\}]_{,j} =$$
$$v_{i,j}\{\delta_{ij}P + \mu(u_{i,j} + u_{j,i})\} + v_i\{\delta_{ij}P + \mu(u_{i,j} + u_{j,i})\}_{,j}$$

$$(v_i\delta_{ij}P)_{,j} = v_{i,j}\delta_{ij}P = v_{i,i}P$$

$$\iiint_\Omega \{v_{i,i}P + v_{i,j}\mu(u_{i,j} + u_{j,i})\}\,dV =$$
$$\iiint_\Omega v_i\rho f_i\,dV + \iiint_\Omega v_i\{\delta_{ij}P + \mu(u_{i,j} + u_{j,i})\}_{,j}\,dV$$

$$\iiint_\Omega \{v_{i,i}P + v_{i,j}\mu(u_{i,j} + u_{j,i})\}\,dV =$$
$$-\iiint_\Omega v_i\rho f_i\,dV + \iint_{\delta\Omega} v_i\sigma_{Ni}\,dA$$

$$\sigma_{Ni} = \{\delta_{ij}P + \mu(u_{i,j} + u_{j,i})\}_N$$